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GENERALIZED B-STRONGLY B*-SEPARATION AXIOMS IN TOPOLOGICAL SPACES

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Abstract
The purpose of this paper is to introduce a new class of spaces via gbsb*-open sets and gbsb*-difference sets. Further we give some basic properties and their various characterizations

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Introduction
D.Andrijevic[1] introduced the concept of b-open sets and characterized its topological properties. Caldas and Jafari [2], introduced and studied b-T_0, b-T_1, b-T_2, b-D_0, b-D_1 and b-D_2 via b-open sets after that Keskin and Noiri [5], introduced the notion of b-$T_{1/2}$. A.Poongothai and P.Parimelazhagan,[6] introduced sb*-closed sets and we extend this concept into gbsb*-open sets[7].

In this chapter, we introduce a new classes of spaces called gbsb*-T_k spaces, for k=0, 1, 2, 1/2, gbsb*-D_k spaces, for k=0,1,2 and gbsb*-spaces. Also we study some basic properties and their various characterizations.

Preliminaries
Throughout this paper (X, τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. (X, τ) will be replaced by X if there is no changes of confusion. For a subset A of a topological space X, cl(A) and int(A) denote the closure of A and the interior of A respectively. We recall the following definitions and results.

Definition 2.1.[1] Let (X, τ) be a topological space. A subset A of the space X is said to be b-open if A⊆int(cl(A))∪cl(int(A)) and b-closed if int(cl(A))∩cl(int(A)) ⊆A.

Definition 2.2. Let (X, τ) be a topological space and A$\subseteq$X. The b-closure of A, denoted by bcl(A) and is defined by the intersection of all b-closed sets containing A.

Definition 2.3.[6] Let (X, τ) be a topological space. A subset Aof X is said to be strongly b*-closed (briefly sb*-closed) if cl(int(A)))$\subseteq$U whenever A$\subseteq$U and U is b-open in (X, τ).

Definition 2.4.[7] A subset A of a topological space (X, τ) is called a generalized b-strongly b*-closed set (briefly,gbsb*-closed) if bcl(A)$\subseteq$U whenever A$\subseteq$U and U is sb*-open in (X, τ). The collection of all gbsb*-closed sets of X is denoted by gbsb*-C(X,τ).

Definition 2.5.[7] The complement of the gbsb*-closed set is a gbsb*-open set. The collection of all gbsb*-open sets of X is denoted by gbsb*-O(X,τ).

Definition 2.6.[8] Let A be a subset of a topological space (X, τ). Then the union of all gbsb*-open sets contained in A is called the gbsb*-interior of A and it is denoted by gbsb*int(A). That is gbsb*int(A)=∪{V:V$\subseteq$A and V$\in$gbsb*-O(X)}.

Definition 2.7.[8] Let A be a subset of a topological space (X, τ). Then the intersection of all gbsb*-closed sets in X containing A is called the gbsb*-closure of A and it is denoted by gbsb*cl(A). That is gbsb*cl(A)=∩{F: A⊆F and F∈gbsb*-C(X, τ)}.

Theorem 2.8.[8] Let A be a subset of a topological space (X, τ). Then

(i) A is gbsb*-open if and only if gbsb*int(A)=A.
(ii) A is gbsb*-closed if and only if gbsb*cl(A)=A.

Theorem 2.9.[7] For every element x in a space X, X\{x} is either gbsb*-closed or sb*-open.

Theorem 2.10.[6] Every closed set is sb*-closed.

Definition 2.11.[7] Let X be a topological space and let x∈X. A subset N of X is said to be a gbsb*-neighbourhood (shortly, gbsb*-nbhd) of x if there exsits a gbsb*-open set U such that x∈U⊆N.

Generalized b-strongly b*-T_k spaces

Definition 3.1. A topological space (X, τ) is said to be

(i) gbsb*-T_0 if for each pair of distinct points x, y in X, there exists a gbsb*-open set U in X such that either x∈U and y∉U ssor x∉U and y∈U.
(ii) gbsb*-T_1 if for each pair of distinct points x, y in X, there exist two gbsb*-open sets U in X and V such that x∈U but y∉U and x∉V and y∈V.
(iii) gbsb*-T_2 if for each pair of distinct points x, y in X, there exist two disjoint gbsb*-open sets U and V in X such that x∈U but y∉U and x∉V and y∈V.
(iv) gbsb*-$T_{1/2}$ if every gbsb*-closed set is sb*-closed.
(v) gbsb*-space if every gbsb*-open set is open.

Theorem 3.2. A topological space (X,τ) is gbsb*-T_0 if and only if for each pair of distinct points x, y in X, gbsb*cl({x})≠gbsb*cl({y}).

Proof: Necessity: Suppose X is gbsb*-T_0 and x,y are any two distinct points of X. Then there exists a gbsb*-open set U containing x or y, say x but not y. Since U is gbsb*-open, X\U is a gbsb*-closed set which does not contain x but contains y. Since gbsb*cl({y}) is the smallest gbsb*-closed set containing y, gbsb*cl({y})⊆X\U. Then x∉gbsb*cl({y}). Hence gbsb*cl({x})≠gbsb*cl({y}).

Sufficiency: Suppose that x,y∈X with x≠y and gbsb*cl({x})≠gbsb*cl({y}). Then there exists a point z∈X such that z∈gbsb*cl({x}) but z∉gbsb*cl({y}). Now, we claim that x∉gbsb*cl({y}). If x∈gbsb*cl({y}), then gbsb*cl({x})⊆gbsb*cl({y}). This implies, z∈gbsb*cl({y}), which contradicts z∉gbsb*cl({y}). Therefore x∉gbsb*cl({y}). Since gbsb*cl({y}) is gbsb*-closed set containing y but not x, then X\gbsb*cl({y}) is a gbsb*-open set containing x but not y. Hence X is a gbsb*-T_0 space.

Theorem 3.3. A topological space (X,τ) is gbsb*-T_1 if and only if the singletons are gbsb*-closed sets.

Proof: Let (X,τ) be a gbsb*-T_1 space and x be any point of X. Let y∈X\{x}. Then x≠y and so there exists a gbsb*-open set U_y containing y but not x. That is y∈U_y⊆X\{x}. This implies, X\{x}=∪{U_y/y∈X\{x}}. Since the union of gbsb*-open sets is gbsb*-open, then X\{x} is gbsb*-open containing y but not x. Hence {x} is gbsb*-closed in X. Conversely, suppose {p} is gbsb*-closed, for every p∈X. Let x,y∈X with x≠y. Then y∈X\{x} and x∈X\{y}. Since {x} and {y} are gbsb*-closed sets in X, then X\{x} and X\{y} are gbsb*-open sets in X. Thus, we have a gbsb*-open set containing x but not y and a gbsb*-open set containing y but not x. Hence X is a gbsb*-T_1 space.

Theorem 3.4. A topological space (X,τ) is gbsb*-$T_{1/2}$ if each singleton {x} of X is either sb*-closed or sb*-open.

Proof: Let (X,τ) is a gbsb*-$T_{1/2}$ space.
Case (i): Suppose {x} is not sb*-closed. Then X\{x} is not sb*-open. By Theorem 2.9, X\{x} is gbsb*-closed. Since X is a gbsb*-$T_{1/2}$ space, then X\{x} is sb*-closed and hence {x} is sb*-open.
Case (ii): Suppose {x} is not sb*-open. Then X\{x} is not sb*-closed. By Theorem 2.9, X\{x} is gbsb*-open in X. Since X is a gbsb*-$T_{1/2}$ space, then X\{x} is sb*-open and hence {x} is sb*-closed.

Theorem 3.5. The following statements are equivalent for a topological space X.
(i) X is gbsb*-T_2.
(ii) For each x∈X and y≠x, there exists a gbsb*-open set U containing x such that y∉gbsb*cl(U).
(iii) For each x∈X, ∩{gbsb*cl(U)/U∈gbsb*-O(X, τ) and x∈U}={x}.

Proof:
(i)⇒(ii): Suppose X is gbsb*-T_2. Then for x,y∈X with x≠y. Then there exists disjoint gbsb*-open sets U and V containing x and y respectively. Since V is gbsb*-open, then X\V is gbsb*-closed containing U. Hence gbsb*cl(U)⊆X\V. Since y∈V, then y∉X\V and hence y∉gbsb*cl(U).

(ii)⇒(iii): If there exists an element y≠x in X such that y∈∩{gbsb*cl(U)/U∈gbsb*-O(X) and x∈U}, then y∈gbsb*cl(U) for every gbsb*-open set U containing x. This contradicts our assumption. So there exists no such an element y. This proves (iii).

(iii)⇒(i): Let x,y∈X with x≠y. Then by our assumption, there exists a gbsb*-open set U containing x such that y∉gbsb*cl(U). Let V=X\gbsb*cl(U). Then V is gbsb*-open set containing y. Also x∈U and U∩V= ϕ. Thus we have a disjoint gbsb*-open sets U and V containing x and y respectively. Hence X is a gbsb*-T_2 space.

Remark 3.6. Every gbsb*-T_2 space is gbsb*-T_1.

Theorem 3.7. Every gbsb*-space is gbsb*-$T_{1/2}$.
Proof: Let (X,τ) be a gbsb*-space and A be any gbsb*-closed set in X. Then X\A is gbsb*-open in X. Since X is gbsb*-space, then X\A is open in X and so A is closed. By Theorem 2.10, A is sb*-closed. Since shows that X is gbsb*-$T_{1/2}$.

Generalized b-strongly b*-D_k spaces

Definition 4.1. A subset A of a topological space X is called a gbsb*-difference set(briefly gbsb*-D-set) if there exists U, V∈gbsb*-O(X) such that U≠X and A=U\V.

Theorem 4.2. Every proper gbsb*-open set is a gbsb*-D-set.

Proof: Let A be any proper gbsb*-open subset of a topological space X. Take U=A and V= φ. Then A=U\V and U≠X. Hence A is gbsb*-D-set.

Remark 4.3. The converse of the above theorem need not be true which is shown in the following example.

Example 4.4. Let X={a,b,c,d} with a topology τ ={φ, {a,b}, {a,b,c}, X}. Then gbsb*-O(X, τ)={φ, {a}, {b}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, X}. Take U={a,b,d} and V={a,b,c}. Then U≠X and A=U\V={a,b,d}\{a,b,c}={d} is gbsb*-D-set but not a gbsb*-open set.

Definition 4.5. A topological space (X, τ) is said to be
(i) gbsb*-D_0 if for any pair of distinct points x and y of X there exists a gbsb*-D-set of X containing x but not y or a gbsb*-D-set of X containing y but not x.

(ii) gbsb*-D_1 if for any pair of distinct points x and y of X there exists a gbsb*-D-set of X containing x but not y and a gbsb*-D-set of X containing y but not x.
(iii) gbsb*-D_2 if for any pair of distinct points x and y of X, there exist disjoint gbsb*-D-sets G and E of X containing x and y respectively.

Theorem 4.6. In a topological space (X, τ),

(i) if (X,τ) is gbsb*-T_k, then it is gbsb*-D_k, for k = 0, 1, 2.
(ii) if (X,τ) is gbsb*-D_k, then it is gbsb*-D_{k-1}, for k = 1, 2.

Proof.

(i) First we prove the result for k=0. Suppose (X, τ) is gbsb*-T_0. Then for each pair of distinct points x, y in X, there exists a gbsb*-open set U such that either $x\in U$ and $y\notin U$ or $y\in U$ and $x\notin U$. By Theorem 4.2, U is gbsb*-D-set in X. Then we have for each pair of distinct points x, y in X, there exists a gbsb*-D-set U such that either $x\in U$ and $y\notin U$ or $y\in U$ and $x\notin U$. Hence (X,τ) is a gbsb*-D_0 space. Similarly we can prove that every gbsb*-T_k space is gbsb*-D_k space, for k=1,2.
(ii) Let k=2. Suppose (X,τ) is a gbsb*-D_2 space. Then for any pair of distinct points x and y of X, there exists disjoint gbsb*-D-sets U and V of X containing x and y respectively. That is for any pair of distinct points x and y of X, there exists a gbsb*-D-set U of X containing x but not y and a gbsb*-D-set V of X containing y but not x. Hence (X,τ) is a gbsb*-D_1 space. Similarly we can prove that every gbsb*-D_1 space is a gbsb*-D_0 space.

Theorem 4.7. A space X is gbsb*-D_0 if and only if it is gbsb*-T_0.

Proof. Necessity: Suppose that X is gbsb*-D_0. Then for each distinct pair x, $y\in X$ there is a gbsb*-D-set G containing x or y, say x but not y. Since G is gbsb*-D-set, then there are two gbsb*-open sets U_1 and U_2 such that $U_1\neq X$ and $G=U_1\setminus U_2$. Since $x\in G$ and $y\notin G$, then $x\in U_1$. For $y\notin G$, we have two cases,

(a) $y\notin U_1$
(b) $y\in U_1$ and $y\in U_2$.

In case (a), $x\in U_1$ and $y\notin U_1$. In case (b), $y\in U_2$ and $x\notin U_2$. Thus in both cases we have for each pair of distinct points x and y in X, there exists a gbsb*-open set U_1 containing x but not y or a gbsb*-open set U_2 containing y but not x. Hence X is gbsb*-T_0.

Suffency: Suppose (X,τ) is gbsb*-T_0. Then by Theorem 4.6(i), (X, τ) is gbsb*-D_0.

Theorem 4.8. A space X is gbsb*-D_1 if and only if it is gbsb*-D_2.

Proof: Necessity: Let x, $y\in X$, with $x\neq y$. Then there exist gbsb*-D -sets G_1,G_2 in X such that $x\in G_1$, $y\notin G_1$ and $y\in G_2$, $x\notin G_2$. Since G_1 and G_2 are gbsb*-D-sets, then $G_1= U_1\setminus U_2$ and $G_2=U_3\setminus U_4$, where U_1, U_2, U_3 and U_4 are gbsb*-open sets in X. From $x\notin G_2$, it follows that either $x\notin U_3$ or $x\in U_3$ and $x\in U_4$. We discuss the two cases separately.

(i)Suppose $x\notin U_3$. For $y\notin G_1$ we have two sub-cases:

(a) Suppose $y\notin U_1$. Since $x\in U_1\setminus U_2$, it follows that $x\in U_1\setminus(U_2\cup U_3)$, and since $y\in U_3\setminus U_4$ we have $y\in U_3\setminus(U_1\cup U_4)$. Since the union of gbsb*-open sets is gbsb*-open set, then $U_2\cup U_3$ and $U_1\cup U_4$ are gbsb*-open sets. Also $(U_1\setminus(U_2\cup U_3))\cap(U_3\setminus(U_1\cup U_4))=\varphi$. Thus we have disjoint gbsb*-D-sets $U_1\setminus(U_2\cup U_3)$ and $U_3\setminus(U_1\cup U_4)$ containing x and y respectively.
(b) If $y\in U_1$ and $y\in U_2$, we have $x\in U_1\setminus U_2$, and $y\in U_2$. Also $(U_1\setminus U_2)\cap U_2 = \varphi$. Thus we have disjoint gbsb*-D-sets $U_1\setminus U_2$ and U_2 containing x and y respectively.

(ii)Suppose $x\in U_3$ and $x\in U_4$. We have $y\in U_3\setminus U_4$ and $x\in U_4$. Hence $(U_3\setminus U_4)\cap U_4= \varphi$. Thus we have disjoint gbsb*-D-sets U_4 and $U_3\setminus U_4$ containing x and y respectively. Hence X is gbsb*-D_2.

Sufficiency: Suppose X is gbsb*-D_2. Then by Theorem 4.6(ii), X is gbsb*-D_1.

Definition 4.9. A point x∈X which has only X as the gbsb*-neighbourhood is called a gbsb*-neat point.

Theorem 4.10. For a gbsb*-T_0 space (X,τ) the following are equivalent:

(i) (X,τ) is gbsb*-D_1.
(ii) (X,τ) has no gbsb*-neat point.

Proof. (i)⇒(ii). Since (X,τ) is gbsb*-D_1, then each point x of X is contained in a gbsb*-D-set A = U\V and thus in U. By definition U≠X. This implies that x is not a gbsb*-neat point.

(ii)⇒(i) Suppose (X,τ) has no gbsb*-neat point. Let x and y be distinct points in X. Since X is gbsb*-T_0, then there exists a gbsb*-open set U containing x or y, say x. Since y∉U, then U≠X. By Theorem 4.2, U is a gbsb*-D-set. Since X has no gbsb*neat point, then y is not a gbsb*-neat point. This means that there exists a gbsb*-neighbourhood V of y such that V≠X. Since V is a gbsb*-nbhd of y, there exists a gbsb*-open set G such that y∈G⊆V. Thus y∈G\U but not x. Also G\U is a gbsb*-D-set. Hence X is a gbsb*-D_1 space.

Corollary 4.11. A gbsb*-T_0 space X is not gbsb*-D_1 if and only if there is a unique gbsb*-neat point in X.
Proof: Suppose (X, τ) be a gbsb*-T_0 space. But (X,τ) is not a gbsb*-D_1. Then by the above theorem (X, τ) has a gbsb*-neat point. Now we have to prove the uniqueness. Suppose x and y are two different gbsb*-neat points in X. Since X is gbsb*-T_0, at least one of x and y, say x, has a gbsb*-open set U containing x but not y. then U is a gbsb*-nbhd of x and U≠X. Therefore x is not a gbsb*-neat point which contradicts x is a gbsb*-neat point. Hence x=y..

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