INTERNATIONAL JOURNAL OF RESEARCH SCIENCE & MANAGEMENT

SOME RESULTS OF ALTERNATING SYSTEMS

Gengrong Zhang[1*], Jiankang Cheng[2], Xiaoting Li[3]
College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi, 530004, China
*Correspondence Author: gxumathzgr@126.com

Keywords: alternating system; pseudo periodic points;

Abstract

In this paper , we talk about the relationship of $[f,g]$ and $[g,f]$. We show that : 1. $\omega([f,g])=\omega([g,f])$, $\omega([f,g])=\omega(g\circ f)\cup\omega(f\circ g)$; 2. $\omega([f,g])\subset$

$\Omega([f,g])\cup\Omega([g,f])$..

Introduction

Let (X,d) be a compact metric space. Denoted by $C^0(X)$ the set of all continuous maps from X to itself. Let $f,g\in C^0(X)$, f^0 is the identity map, $f^{n+1}=$

$f\circ f^n$. Denoted by $g\circ f$ the composition of g and f. Let Z_+ be the set of all positive integers, $N=\{0\}\cup Z_+$. Denoted by $\#\{A\}$ the cardinality of A, $\aleph_0$ $=\#\{N\}$. Let $\overline{A}$ be the closure of A.

Let $(X,[f,g])$ be the alternating systems (see [1] for the definition). For any x $\in X$, write $O(x,[f,g])=\{x,f(x),gf(x),fgf(x),\cdots\}$, we call $O(x,[f,g])$ the orbit of x under $[f,g]$. For any $n\in N$, write

$$F_n(x)=\begin{cases}(gf)^k & , \quad n=2k\\ \left(f(gf)^k\right)(x) & , \quad n=2k+1\end{cases}$$

So sometimes we can replace $[f,g]$ by $\{F_n\}_{n\in N}$, therefore the alternating system $(X,[f,g])=(X,\{F_n\}_{n\in N})$ and $O(x,[f,g])=\{F_n(x)\}_{n\in N}$.

Obviously , $O(x,[f,g])=\{x,f(x),gf(x),fgf(x),\cdots\}$

$$=O(x,g\circ f)\cup O(f(x),f\circ g)$$

$$=O(x,g\circ f)\cup f(O(x,g\circ f))$$

Similarly , write

$$G_n(x)=\begin{cases}(fg)^k & , \quad n=2k\\ \left(g(fg)^k\right)(x) & , \quad n=2k+1\end{cases}$$

The orbit of x under f, the set of periodic points under f, the set of recurrent points under f, the set of non-wondering points under f, the ω- limit set of x un-
der f will be denoted by $O(x,f)$, $P(f)$, $R(f)$, $\Omega(f)$, $\omega(x,f)$, respectively (see [3] for the detailed definitions).

 ISSN: 2349- 5197

INTERNATIONAL JOURNAL OF RESEARCH SCIENCE & MANAGEMENT

2. Definitions and lemmas

Definition 2.1 Denoted by $\omega(x,[f,g])$ the set of all limit points of $\{F_n(x)\}_{n\in N}$, write $\omega([f,g])=\bigcup_{x\in X}\omega(x,[f,g])$.

Definition2.2 Define the set of pseudo periodic points under alternating system $\tilde{P}([f,g])=\{x\in X|x\in\omega(x,[f,g]),\#\{\omega(x,[f,g])\}<\aleph_0\}$.

Definition 2.3 Define the set of recurrent points under alternating system $R([f,g])=\{x\in X|x\in\omega(x,[f,g])\}$.

Obviously, $\tilde{P}([f,g])\subset R([f,g])\subset\omega([f,g])$.

Definition 2.4 A point $x\in X$ is said to be non-wondering under alternating system if for any neighborhood U of x there exists $n\in Z_+$ such that $F_n(U)\cap U\neq\phi$. Denoted by $\Omega([f,g])$ the set of all non-wondering points under alternating system.

Lemma 2.5[2] Let (X,d) be a compact metric space, $f,g\in C^0(X)$, then

(1) For any $n\in Z_+$, we have $F_n=G_{n-1}\circ f$, $G_n=F_{n-1}\circ g$.

(2) If n is even, then $F_{n+t}=F_t\circ F_n$.

(3) If n is odd, then $F_{n+t}=G_t\circ F_n$.

Lemma 2.6[2] $\omega(x,[f,g])=\omega(x,g\circ f)\cup\omega(f(x),f\circ g)$.

Lemma 2.7[2] $\omega(f(x),f\circ g)=f(\omega(x,g\circ f))$.

Lemma 2.8[1] $\omega(x,f)\neq\phi,\overline{\omega(x,f)}=\omega(x,f)$.

Lemma 2.9[1] (1) $\tilde{P}([f,g])\subset P(g\circ f)\cup P(f\circ g)$;

(2) $P(g\circ f)\subset\tilde{P}([f,g])$;

(3) $g(P(f\circ g))\subset\tilde{P}([f,g])$;

(4) $f(P(g\circ f))=P(f\circ g)$.

Lemma 2.10[4] Let f be a continuous map of compact interval to itself. If the set of periodic points of f is a closed set, then every chain recurrent point is periodic.

Lemma 2.11[2] $\omega([f,g])\subset\omega(g\circ f)\cup\omega(f\circ g)$.

3. Main results

Proposition 3.1 $\omega(x,[f,g])\neq\phi$, $\overline{\omega(x,[f,g])}=\omega(x,[f,g])$.

Proof: Combine Lemma2.6 with Lemma2.8, it is trivial.

Proposition 3.2 $\omega(x,[f,g])=\bigcap_{n\geq 0}\overline{\{F_n(x),F_{n+1}(x),F_{n+2}(x),\cdots\}}$.

Proof: We firstly prove $\omega(x,[f,g])\subset\bigcap_{n\geq 0}\overline{\{F_n(x),F_{n+1}(x),F_{n+2}(x),\cdots\}}$. For any $y\in\omega(x,[f,g])$, there exists $m_k\to\infty$ such that $F_{m_k}(x)\to y$, then for any $n\in N$, there exists $k\in Z_+$ such that .$m_k\geq n$, hence

$y\in\bigcap_{n\geq 0}\overline{\{F_n(x),F_{n+1}(x),F_{n+2}(x),\cdots\}}$.

INTERNATIONAL JOURNAL OF RESEARCH SCIENCE & MANAGEMENT

Now we prove $\omega(x,[f,g]) \supset \bigcap_{n\geq 0}\overline{\bigcup_{k=n}^{\infty}\{F_k(x)\}}$. For any $y \in \bigcap_{n\geq 0}\overline{\bigcup_{k=n}^{\infty}\{F_k(x)\}}$, suppose $y \notin \omega(x,[f,g])$, there exists a neighborhood U of y such that for any $n \in N$, $F_n(x) \notin U$, which is contrary to the hypothesis that for any y, we have $y \in \bigcap_{n\geq 0}\overline{\{F_n(x), F_{n+1}(x), F_{n+2}(x), \cdots\}}$.

Hence $y \in \omega(x,[f,g])$.

Proposition 3.3 (1) If n is even, then $\omega(x,[f,g]) = \omega(F_n(x),[f,g])$;

(2) If n is odd, then $\omega(x,[f,g]) = \omega(F_n(x),[g,f])$.

Proof : (1) Suppose that n is even. We prove $\omega(x,[f,g]) \subset \omega(F_n(x),[f,g])$ firstly. If $y \in \omega(x,[f,g])$, there exists $n_k \geq n$, $n_k \to \infty$ such that $F_{n_k}(x) \to y$, by Lemma 2.5 (2), we have $F_{n_k - n}(F_n(x)) \to y$, hence $y \in \omega(F_n(x),[f,g])$.

Now we prove $\omega(x,[f,g]) \supset \omega(F_n(x),[f,g])$. If $z \in \omega(F_n(x),[f,g])$, there exists $m_k \to \infty$ such that $F_{m_k}(F_n(x)) \to z$, by Lemma 2.5 (2), we have $F_{m_k+n}(x) \to z$. Hence $z \in \omega(x,[f,g])$.

(2) Similar to the proof of (1) , by Lemma 2.5 (3) , the proof of (2) is trivial.

Theorem 3.4 $\omega([f,g]) = \omega([g,f])$.

Proof : We prove $\omega([f,g]) \subset \omega([g,f])$ only. For any $x \in \omega([f,g])$, there exists $y \in X$ and $n_k \to \infty$ such that $F_{n_k}(y) \to x$, by Lemma2.5(1), we have $(G_{n_{k-1}} \circ f)(y)$ $= G_{n_{k-1}}(f(y)) \to x$, thus $x \in \omega([g,f])$.

Proposition 3.5 $\tilde{P}([f,g]) \cup \tilde{P}([g,f]) = P(g \circ f) \cup P(f \circ g)$.

Proof : By Lemma2.9(1), we have $\tilde{P}([f,g]) \subset P(g \circ f) \cup P(f \circ g)$, exchange f,g, then $\tilde{P}([g,f]) \subset P(g \circ f) \cup P(f \circ g)$, hence

$\tilde{P}([f,g]) \cup \tilde{P}([g,f]) \subset P(g \circ f) \cup P(f \circ g)$ (1)

By Lemma2.9 (2), we have $P(g \circ f) \subset \tilde{P}([f,g])$, exchange f,g, then $P(f \circ g) \subset \subset \tilde{P}([g,f])$, hence

$P(g \circ f) \cup P(f \circ g) \subset \tilde{P}([f,g]) \cup \tilde{P}([g,f])$ (2)

Combine (1) with (2) , we have

$P(g \circ f) \cup P(f \circ g) \subset \tilde{P}([f,g]) \cup \tilde{P}([g,f]) \subset P(g \circ f) \cup P(f \circ g)$.

Hence $\tilde{P}([f,g]) \cup \tilde{P}([g,f]) = P(g \circ f) \cup P(f \circ g)$.

Theorem 3.6 (1) $R([f,g]) \subset \Omega([f,g])$;

(2) $\omega(g \circ f) \subset \Omega(g \circ f) \subset \Omega([f,g])$;

(3) $f(\Omega(g \circ f)) \subset \Omega(f \circ g)$;

(4) $\Omega([f,g])$ is closed;

INTERNATIONAL JOURNAL OF RESEARCH SCIENCE & MANAGEMENT

(5) $\Omega([f,g]) \subset \Omega(g \circ f) \cup \Omega(f \circ g)$

Proof : (1) and (2) are evident by definitions.

(3) For any $\varepsilon > 0$, by the continuity of f, there exists $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\varepsilon)$. For any $x \in \Omega(g \circ f)$, By the definition , for $\delta > 0$, there exists $y \in B(x,\delta)$ and $n \in Z_+$ such that $(g \circ f)^n(y) \in B(x,\delta)$. So $f(y) \in B(f(x),\varepsilon)$, $f(g \circ f)^n(y) = (f \circ g)^n(f(y)) \in B(f(x),\varepsilon)$.Hence $x \in \Omega(f \circ g)$.

(4) For any $x \in X - \Omega([f,g])$, by the definition, there exists some neighborhood U of x such that for any $n \in Z_+, F_n(U) \cap U = \phi$. Thus, for any $y \in U, y \in X - \Omega([f,g])$. So $X - \Omega([f,g])$ is open . Hence $\Omega([f,g])$ is closed.

(5) Suppose $x \in \Omega([f,g])$, U is an arbitrary neighborhood of x. If the set $\{n \in Z_+ : F_n(U) \cap U \neq \phi, n = 2k \text{ for some } k \in Z\}$ is infinite, there exist even numbers $n_1 < n_2$ and $y \in U$ such that $(g \circ f)^{n_1}(y) \in U, (g \circ f)^{n_2}(y) \in U$, then $(g \circ f)^{n_2}(y) = (g \circ f)^{n_2 - n_1}((g \circ f)^{n_1}(y)) \in U$. Hence $x \in \Omega(g \circ f)$.

If the set $\{n \in Z_+ : F_n(U) \cap U \neq \phi, \ n = 2k+1 \text{ for some } k \in Z\}$ is infinite, there exist odd numbers $n_1 < n_2$ and $y \in U$ such that $f(g \circ f)^{n_1}(y) = (f \circ g)^{n_1}(f(y)) \in U$ and $f(g \circ f)^{n_2}(y) \in U$, then $f(g \circ f)^{n_2}(y) = (f \circ g)^{n_2 - n_1}((f \circ g)^{n_1}(f(y))) \in U$. Hence $x \in \Omega(f \circ g)$.

In conclusion , $\Omega([f,g]) \subset \Omega(g \circ f) \cup \Omega(f \circ g)$.

Corollary 3.7 $\Omega([f,g]) \cup \Omega([g,f]) = \Omega(g \circ f) \cup \Omega(f \circ g)$.

Proof : Similar to the proof of Proposition 3.5 , by Theorem 3.6 (2) (5), it is evident.

Lemma 3.8 $\omega([f,g]) = \omega(g \circ f) \cup \omega(f \circ g)$.

Proof : By Lemma 2.6, we have $\omega(g \circ f) \subset \omega([f,g])$. By Lemma 2.11, we have $\omega([f,g]) \subset \omega(g \circ f) \cup \omega(f \circ g)$. So it remains to show that $\omega(f \circ g) \subset \omega([f,g])$.

For any $x \in \omega(f \circ g)$, there exists $y \in X$ and $n_k \to \infty$ such that $(f \circ g)^{n_k}(y) \to x$, so we have $f(g \circ f)^{n_k - 1}(g(y)) \to x$. Hence $x \in \omega([f,g])$.

Similarly, $\omega(g \circ f) \subset \omega([f,g])$. Hence $\omega([f,g]) \supset \omega(g \circ f) \cup \omega(f \circ g)$.

Theorem 3.9 $\omega([f,g]) \subset \Omega([f,g]) \cup \Omega([g,f])$.

Proof : By Lemma 3.8 , we have $\omega([f,g]) = \omega(g \circ f) \cup \omega(f \circ g)$. Thus $\omega([f,g])$ $\omega(g \circ f) \cup \omega(f \circ g) \subset \Omega(g \circ f) \cup \Omega(f \circ g) \subset \Omega([f,g]) \cup \Omega([g,f])$.

INTERNATIONAL JOURNAL OF RESEARCH SCIENCE & MANAGEMENT

Proposition 3.10 If $X=[0,1]$, $\widetilde{P}([f,g])\cap\widetilde{P}([g,f])=\phi$, $\widetilde{P}([f,g])$ and $\widetilde{P}([g,f])$ are closed , then $\widetilde{P}([f,g])\cup\widetilde{P}([g,f])=\Omega([f,g])\cup\Omega([g,f])$.

Proof : If $\widetilde{P}([f,g])\cap\widetilde{P}([g,f])=\phi$, then by Lemma 2.9 (2), we have $P(g\circ f)\cap P(f\circ g)=\phi$. Science $\widetilde{P}([f,g])$ and $\widetilde{P}([g,f])$ are closed, by Theorem 3.5, we have $\widetilde{P}([f,g])=P(g\circ f)=\overline{P(g\circ f)}$ and $\widetilde{P}([g,f])=P(f\circ g)=\overline{P(f\circ g)}$. By Lemma 2.10, $\widetilde{P}([f,g])=P(g\circ f)=\Omega(g\circ f)$ and $\widetilde{P}([g,f])=P(f\circ g)=\Omega(f\circ g)$. It follows by Corollary 3.7 that $\widetilde{P}([f,g])\cup\widetilde{P}([g,f])=\Omega([f,g])\cup\Omega([g,f])$.

Corollary 3.11 If $X=[0,1]$, $\widetilde{P}([f,g])\cap\widetilde{P}([g,f])=\phi$, $\widetilde{P}([f,g])$ and $\widetilde{P}([g,f])$ are closed, then $\widetilde{P}([f,g])\cup\widetilde{P}([g,f])=R([f,g])\cup R([g,f])=\omega([f,g])=\Omega([f,g])\cup\Omega([g,f])$.

Proof : Combine Theorem 3.4 , Theorem 3.9 and Proposition 3.10, it is evident.

Example 3.12 Let $X=[-1,1]$, $a_n=1-\dfrac{1}{n+1}$, $n=1,2,3\cdots$. Define

(1) $f(x)=x$, for any $x\in[0,1]$;

(2) $f(-1)=1$, $f(-a_n)=a_{n+1}$, $n=1,2,3\cdots$ and f is linear on $[-a_1,0]$ and each $[-a_{n+1},-a_n]$, $n=1,2,3\cdots$;

(3) $g(x)=-x$, for any $x\in[0,1]$;

(4) $g(x)=x$, for any $x\in[-1,1]$.

It is easy to show that $P(g\circ f)=\omega(g\circ f)=\omega(0,g\circ f)=\Omega(g\circ f)=\{-1\}$; $P(f\circ g)=\omega(f\circ g)=\omega(0,f\circ g)=\Omega(f\circ g)=\{1\}$; For any $x\in X$, $\omega(x,[f,g])=\omega(x,[g,f])=\Omega([f,g])=\Omega([g,f])=\{-1,1\}$.

Proposition 3.13 Let $\Gamma=\omega$ *or* Ω, there exists f and g such that $\Gamma(g\circ f)$ is a proper subset of $\Gamma([f,g])$.

Proof : By Example 3.12, $\omega(g\circ f)=\{-1\}$ and $\omega([f,g])=\bigcup_{x\in X}\omega(x,[f,g])=\{-1,1\}$. Hence $\omega(g\circ f)$ is a proper subset of $\omega([f,g])$. Similarly, $\Omega(g\circ f)$ is a proper subset of $\Omega([f,g])$.

Proposition 3.14 There exists f and g such that $P(g\circ f)$ is a proper subset of $\widetilde{P}([f,g])$.

Proof : By Example 3.12, $P(g\circ f)=\{-1\}$ and $\widetilde{P}([f,g])=\{-1,1\}$. Hence $P(g\circ f)$ is a proper subset of $\widetilde{P}([f,g])$.

Proposition 3.15 There exists f and g such that $\Omega([f,g])=\Omega([g,f])$.

Proof : By Example 3.12 , $\Omega([f,g])=\Omega([g,f])=\{-1,1\}$.

Proposition 3.16 There exists f and g such that $\Omega(f\circ g)$ is a proper subset of $\Omega([f,g])$.

Proof : By Example 3.12, $\Omega(f\circ g)=\{1\}$ and $\Omega([f,g])=\{-1,1\}$. Hence $\Omega(f\circ g)$ is a proper subset of $\Omega([f,g])$.

INTERNATIONAL JOURNAL OF RESEARCH SCIENCE & MANAGEMENT

Example 3.17 Let $X=\{-1,0,1\}\subset[-1,1]$, define $f(-1)=0,\ f(0)=1,\ f(1)=g(1)=1,\ g(0)=-1, g(-1)=-1$.

It is easy to show that (1) $\omega(-1,[f,g])=\{-1,0\}, \omega(0,[f,g])=\omega(1,[f,g])=\{1\}$.
(2) $\Omega(f\circ g)=\{0,1\}, \Omega([f,g])=\{-1,1\}$.

Proposition 3.18 There exists f and g such that $P(f\circ g)\not\subset \tilde{P}([f,g])$.
Proof : By Example 3.12, $P(f\circ g)=\{0,1\}$ and $\tilde{P}([f,g])=\{-1,1\}$. Hence $P(f\circ g)\not\subset \tilde{P}([f,g])$.

Proposition 3.19 There exists f and g such that $\Omega(f\circ g)\not\subset \Omega([f,g])$.
Proof : By Example 3.12, $\Omega(f\circ g)=\{0,1\}$ and $\Omega([f,g])=\{-1,1\}$. Hence $\Omega(f\circ g)\not\subset \Omega([f,g])$.

Proposition 3.20 There exists f and g such that $\omega([f,g])\not\subset \Omega([f,g])$.
Proof : By Example 3.12, $\omega([f,g])=\bigcup_{x\in X}\omega(x,[f,g])=\{-1,0,1\}$ and $\Omega([f,g])=\{-1,1\}$. Hence $\Omega(f\circ g)\not\subset \Omega([f,g])$.

Acknowledgment
Project supported by NSFC(No: 11461002) and NSF of Guangxi Education Department(No: 2013ZD061).

References
1. D'Aniello E, Steele T H. The ω-limit sets of alternating systems[J]. Journal of Difference Equations and Applications, 2011, 17(12): 1793-1799.
2. Hailan Liang. Some dynamical properties of non-autonomous system. Master thesis: Guangxi University; 2012.
3. Gongfu Liao, Lidong Wang, Qinjie Fan. Iterate and chaos dynamic system [M].Science publication.
4. Block L, Franke J E. The chain recurrent set for maps of the interval[J]. Proceedings of the American Mathematical Society, 1983, 87(4): 723-727.