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A COMMON FIXED POINT THEOREM FOR UNIQUE RANDOM POINT IN HILBERT SPACE USING SIX RANDOM OPERATORS

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Keywords: Common fixed point, random operators, weakly compatible, semi-compatible, Separable Hilbert Space.
AMS subject classification (2000): 47H10, 54H25.

Abstract
The aim of this paper is to obtain a common fixed point theorem for six random operators by using weak compatibility, semi-compatibility in the non- empty closed subset of a separable Hilbert Space. Our results generalize and extend the result.

Introduction
The study of random fixed point theory has attracted much attention in recent years [4-6] and [9, 11]. Badshah and Sayyed [2], Badshah and Gagrani [1], have proved various common random fixed point theorem in Polish space. Choudhury [7] construct a sequence of measurable function and consider its convergence to find a common unique fixed point of two random operators in Hilbert space. Badshah and Shrivastava [3] introduced the concept of semi-compatibility in Polish spaces. These results extend the corresponding result in [10].

In this paper we construct a sequence of measurable functions and consider its convergence to the common unique random fixed point of six continuous random operators defined on a non-empty closed subset of a Separable Hilbert space.

Preliminary notes
Let C be a closed subset of Separable Hilbert space H and (Ω, Σ) a measurable space.
Definition 2.1: A function $f: \Omega \to C$ is called measurable if $f^{-1}(B \cap C) \in \Sigma$ for each Borel subset B of H.
Definition 2.2: A function $F: \Omega \times C \to C$ is called random operator if $F(., x): \Omega \to C$ is measurable for all $x \in C$.
Definition 2.3: A measurable function $g: \Omega \to C$ is called a random fixed point to the random operator $F: \Omega \times C \to C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.
Definition 2.4: A random operator $F: \Omega \times C \to C$ is called continuous if for fixed $t \in \Omega$, $F(t, .): C \to C$ is continuous.
Definition 2.5: Two mappings $f, g: X \to X$ where X is a Polish space, are called compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, provided that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ exist in X.
Definition 2.6: Two random operators $E, F: \Omega \times X \to X$ are called compatible if $E(t, .)$ and $F(t, .)$ are compatible for all $t \in \Omega$.
Definition 2.7: Two random operators $E, F: \Omega \times X \to X$ are called weakly compatible if $E(t, g(t)) = F(t, g(t))$ for some measurable mapping $g: \Omega \to X$

$$E\left(t, F(t, g(t))\right) = F\left(t, E(t, g(t))\right), \text{ For all } t \in \Omega.$$

Definition 2.8: Let $g_n: \Omega \to X$ is a measurable mapping such that $E(t, g_n(t)), F(t, g_n(t)) \to g(t)$ as $n \to \infty$ for some measurable mapping $g: \Omega \to X$, then random operators $E, F: \Omega \times X \to X$ are called semi- compatible if $d\left(E\left(t, F(t, g_n(t))\right), F(t, g(t))\right) \to 0$ as $n \to \infty$.

Main results
Theorem 3.1: Let C be a non-empty closed subset of a Separable complete Hilbert space H. Let E, F, P, Q R and S be the six continuous random operators defined on C such that for $t \in \Omega$, $E(t,.), F(t,.), P(t,.), Q(t,.), R(t,.), S(t,.): \Omega \times C \to C$ satisfy the following Conditions

(1) $EF(t, X) \subset Q(t, X)$ and $RS(t, X) \subset P(t, X)$
(2) $$\|EF(t,x) - RS(t,y)\|^2 \le \alpha(t)\|P(t,x) - EF(t,x)\|^2 + \beta(t)\|Q(t,y) - RS(t,y)\|^2 + \gamma(t)\|P(t,x) - Q(t,y)\|^2$$

For all $x, y \in C, t \in \Omega$ where $\alpha(t), \beta(t), \gamma(t): \Omega \to (0,1)$ are measurable mapping such that $\alpha(t) + \beta(t) + \gamma(t) < 1$.

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(3) If either (i) or (ii)

1. P or EF is continuous and (RS, Q) are weakly compatible, (EF, P) are semi-compatible.
2. Q or RS is continuous and (EF, P) are weakly compatible, (RS, Q) are semi-compatible.

Then EF, RS, P and Q have a unique common random fixed point in C.

Proof: Let $g_0: \Omega \to C$ be an arbitrary measurable mapping and $g_n: \Omega \to C$ be a sequence of measurable mappings such that
$y_{2n}(t)=EF(t, g_{2n}(t)) = Q(t, g_{2n+1}(t))$,
$y_{2n+1}(t)=RS(t, g_{2n+1}(t)) = P(t, g_{2n+2}(t))$.
For all $t \in \Omega$ and $n = 0, 1, 2, \dots\dots\dots$

$$\|y_{2n}(t) - y_{2n+1}(t)\|^2 = \|EF(t, g_{2n}(t)) - RS(t, g_{2n+1}(t))\|^2$$
$$\leq \alpha(t)\|P(t, g_{2n}(t)) - EF(t, g_{2n}(t))\|^2 + \beta(t)\|Q(t, g_{2n+1}(t)) - RS(t, g_{2n+1}(t))\|^2 + \gamma(t)\|P(t, g_{2n}(t)) - Q(t, g_{2n+1}(t))\|^2$$
$$\leq \alpha(t)\|y_{2n-1}(t) - y_{2n}(t)\|^2 + \beta(t)\|y_{2n}(t) - y_{2n+1}(t)\|^2 + \gamma(t)\|y_{2n-1}(t) - y_{2n}(t)\|^2$$
$$\Rightarrow (1 - \beta(t))\|y_{2n}(t) - y_{2n+1}(t)\|^2 \leq (\alpha(t) + \gamma(t))\|y_{2n-1}(t) - y_{2n}(t)\|^2$$
$$\Rightarrow \|y_{2n}(t) - y_{2n+1}(t)\|^2 \leq \left(\frac{\alpha(t) + \gamma(t)}{(1 - \beta(t))}\right)\|y_{2n-1}(t) - y_{2n}(t)\|^2$$
$$\Rightarrow \|y_{2n}(t) - y_{2n+1}(t)\|^2 \leq k(t)\|y_{2n-1}(t) - y_{2n}(t)\|^2$$

Where $k(t) = \left(\frac{\alpha(t)+\gamma(t)}{(1-\beta(t))}\right) < 1$.

And,

$$\|y_{2n+1}(t) - y_{2n+2}(t)\|^2 = \|EF(t, g_{2n+1}(t)) - RS(t, g_{2n+2}(t))\|^2$$
$$\leq \alpha(t)\|P(t, g_{2n+1}(t)) - EF(t, g_{2n+1}(t))\|^2$$
$$+\beta(t)\|Q(t, g_{2n+2}(t)) - RS(t, g_{2n+2}(t))\|^2$$
$$+\gamma(t)\|P(t, g_{2n+1}(t)) - Q(t, g_{2n+2}(t))\|^2$$
$$\leq \alpha(t)\|y_{2n}(t) - y_{2n+1}(t)\|^2 + \beta(t)\|y_{2n+1}(t) - y_{2n+2}(t)\|^2 + \gamma(t)\|y_{2n}(t) - y_{2n+1}(t)\|^2$$
$$\Rightarrow (1 - \beta(t))\|y_{2n+1}(t) - y_{2n+2}(t)\|^2 \leq (\alpha(t) + \gamma(t))\|y_{2n}(t) - y_{2n+1}(t)\|^2$$
$$\Rightarrow \|y_{2n+1}(t) - y_{2n+2}(t)\|^2 \leq \left(\frac{\alpha(t) + \gamma(t)}{(1 - \beta(t))}\right)\|y_{2n}(t) - y_{2n+1}(t)\|^2$$
$$\Rightarrow \|y_{2n+1}(t) - y_{2n+2}(t)\|^2 \leq k(t)\|y_{2n}(t) - y_{2n+1}(t)\|^2$$

Therefore,

$$\|y_{2n+1}(t) - y_{2n+2}(t)\|^2 \leq k^2(t)\|y_{2n-1}(t) - y_{2n}(t)\|^2$$

Proceeding in this manner we get a sequence of measurable mappings $y_{2n}: \Omega \to C$ such that

$$\|y_{2n}(t) - y_{2n+1}(t)\|^2 \leq k^{2n}(t)\|y_0(t) - y_1(t)\|^2$$

Thus for all $t \in \Omega$, $\{y_{2n}(t)\}$ is a Cauchy sequence.
Hence $\{y_{2n}(t)\}$ is convergent in Separable Hilbert space.
Therefore for all $t \in \Omega$, $\{y_{2n}(t)\} \to g(t)$ as $n \to \infty$
$EF(t, g_{2n}(t)) \to g(t)$, $Q(t, g_{2n+1}(t)) \to g(t)$,
$RS(t, g_{2n+1}(t)) \to g(t)$, $P(t, g_{2n+2}(t)) \to g(t)$; For all $t \in \Omega$.

Case I: If P is continuous.
Then we have

$$P(t, EF(t, g_{2n}(t))) \to P(t, g(t)), \text{ And}$$
$$P(t, P(t, g_{2n+2}(t))) \to P(t, g(t)).$$

Since EF and P are semi-compatible.
Therefore, for all $t \in \Omega$,

$$EF(t, P(t, g_{2n}(t))) \to P(t, g(t)).$$

Step I: For all $t \in \Omega$,

$$\|EF(t, P(t, g_{2n}(t))) - RS(t, g_{2n+1}(t))\|^2 \leq \alpha(t)\left\|P(t, P(t, g_{2n}(t))) - EF\left(t, P(t, g_{2n}(t))\right)\right\|^2$$
$$+\beta(t)\|Q(t, g_{2n+1}(t)) - RS(t, g_{2n+1}(t))\|^2$$
$$+\gamma(t)\|P(t, P(t, g_{2n}(t))) - Q(t, g_{2n+1}(t))\|^2$$

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Taking $n \to \infty$, we have

$$\|P(t,g(t)) - g(t)\|^2 \le \alpha(t)\|P(t,g(t)) - P(t,g(t))\|^2 + \beta(t)\|g(t) - g(t)\|^2 + \gamma(t)\|P(t,g(t)) - g(t)\|^2$$
$$\Rightarrow (1-\gamma(t))\|P(t,g(t)) - g(t)\|^2 \le 0$$
$$\Rightarrow P(t,g(t)) = g(t).$$

Step II: For all $t \in \Omega$,

$$\|EF(t,g(t)) - RS(t,g_{2n+1}(t))\|^2 \le \alpha(t)\|P(t,g(t)) - EF(t,g(t))\|^2$$
$$+\beta(t)\ \|Q(t,g_{2n+1}(t)) - RS(t,g_{2n+1}(t))\|^2$$
$$+\gamma(t)\|P(t,g(t)) - Q(t,g_{2n+1}(t))\|^2$$

Taking $n \to \infty$, we have

$$\|EF(t,g(t)) - g(t)\|^2 \le \alpha(t)\|g(t) - EF(t,g(t))\|^2 + \beta(t)\|g(t) - g(t)\|^2 + \gamma(t)\|g(t) - g(t)\|^2$$
$$\Rightarrow (1-\alpha(t))\|EF(t,g(t)) - g(t)\|^2 \le 0$$
$$\Rightarrow EF(t,g(t)) = g(t).$$

Hence

$$EF(t,g(t)) = g(t) = P(t,g(t)).$$

Since, $EF(t,X) \subset Q(t,X)$.
Therefore, there exist a measurable mapping $\xi : \Omega \to C$ such that

$$EF(t,g(t)) = Q(t,\xi(t))$$

Therefore,

$$g(t) = EF(t,g(t)) = P(t,g(t)) = Q(t,\xi(t))$$

Step III: For all $t \in \Omega$,

$$\|EF(t,g_{2n}(t)) - RS(t,\xi(t))\|^2$$
$$\le \alpha(t)\|P(t,g_{2n}(t)) - EF(t,g_{2n}(t))\|^2 + \beta(t)\|Q(t,\xi(t)) - RS(t,\xi(t))\|^2$$
$$+\ \gamma(t)\|P(t,g_{2n}(t)) - Q(t,\xi(t))\|^2$$

Taking $n \to \infty$, we have

$$\|g(t) - RS(t,\xi(t))\|^2 \le \alpha(t)\|g(t) - g(t)\|^2 + \beta(t)\|g(t) - RS(t,\xi(t)\|^2 + \gamma(t)\|g(t) - g(t)\|^2$$
$$\Rightarrow (1-\beta(t))\|g(t) - RS(t,\xi(t)\|^2 \le 0$$
$$\Rightarrow RS(t,\xi(t)) = g(t).$$

Hence

$$RS(t,\xi(t)) = g(t) = Q(t,\xi(t)).$$

Since, RS and Q are weakly compatible.
Therefore, for all $t \in \Omega$

$$RS\left(t, Q(t,\xi(t))\right) = Q\left(t, RS(t,\xi(t))\right);$$
$$RS(t,g(t)) = Q(t,g(t)).$$

Step IV: For all $t \in \Omega$,

$$\|EF(t,g(t)) - RS(t,g(t))\|^2$$
$$\le \alpha(t)\|P(t,g(t)) - EF(t,g(t))\|^2 + \beta(t)\|Q(t,g(t)) - RS(t,g(t))\|^2 + \gamma(t)\|P(t,g(t)) - Q(t,g(t))\|^2$$

Taking $n \to \infty$, we have

$$\|g(t) - Q(t,g(t))\|^2 \le \alpha(t)\|g(t) - g(t)\|^2 + \beta(t)\|Q(t,g(t)) - Q(t,g(t)\|^2 \qquad + \gamma(t)\|g(t) - Q(t,g(t))\|^2$$
$$\Rightarrow (1-\gamma(t))\|g(t) - Q(t,g(t)\|^2 \le 0$$
$$\Rightarrow Q(t,g(t)) = g(t).$$

Thus for all $t \in \Omega$,

$$EF(t,g(t)) = RS(t,g(t)) = P(t,g(t)) = Q(t,g(t)) = g(t).$$

Hence $g(t)$ is a common random fixed point of EF, RS, P and Q.

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Similarly, we can proof that for Q is continuous.
Case II: If EF is continuous.
Then we have

$$EF(t, EF(t, g_{2n}(t))) \to EF(t, g(t)), \quad \text{And}$$
$$P(t, P(t, g_{2n+2}(t))) \to P(t, g(t)).$$

Since EF and P are semi-compatible.
Therefore, for all $t \in \Omega$,

$$EF(t, P(t, g_{2n}(t))) \to P(t, g(t)).$$

Step I: For all $t \in \Omega$,

$$\|EF(t, P(t, g_{2n}(t))) - RS(t, g_{2n+1}(t))\|^2 \leq \alpha(t) \left\|P(t, P(t, g_{2n}(t))) - EF\left(t, P\left(t, g_{2n}(t)\right)\right)\right\|^2$$
$$+\beta(t)\left\|Q\left(t, g_{2n+1}(t)\right) - RS\left(t, g_{2n+1}(t)\right)\right\|^2$$
$$+\gamma(t)\left\|P(t, P(t, g_{2n}(t))) - Q\left(t, g_{2n+1}(t)\right)\right\|^2$$

Taking $n \to \infty$, we have

$$\left\|P\left(t, g(t)\right) - g(t)\right\|^2 \leq \alpha(t)\left\|P\left(t, g(t)\right) - P\left(t, g(t)\right)\right\|^2 + \beta(t)\|g(t) - g(t)\|^2 + \gamma(t)\left\|P\left(t, g(t)\right) - g(t)\right\|^2$$
$$\Rightarrow \left(1 - \gamma(t)\right)\left\|P\left(t, g(t)\right) - g(t)\right\|^2 \leq 0$$
$$\Rightarrow P\left(t, g(t)\right) = g(t).$$

Step II: For all $t \in \Omega$,

$$\left\|EF\left(t, g(t)\right) - RS\left(t, g_{2n+1}(t)\right)\right\|^2 \leq \alpha(t)\left\|P\left(t, g(t)\right) - EF\left(t, g(t)\right)\right\|^2$$
$$+\beta(t)\left\|Q\left(t, g_{2n+1}(t)\right) - RS\left(t, g_{2n+1}(t)\right)\right\|^2$$
$$+\gamma(t)\left\|P\left(t, g(t)\right) - Q\left(t, g_{2n+1}(t)\right)\right\|^2$$

Taking $n \to \infty$, we have

$$\left\|EF\left(t, g(t)\right) - g(t)\right\|^2 \leq \alpha(t)\left\|g(t) - EF\left(t, g(t)\right)\right\|^2 + \beta(t)\|g(t) - g(t)\|^2 + \gamma(t)\|g(t) - g(t)\|^2$$
$$\Rightarrow \left(1 - \alpha(t)\right)\left\|EF\left(t, g(t)\right) - g(t)\right\|^2 \leq 0$$
$$\Rightarrow EF\left(t, g(t)\right) = g(t).$$

Hence

$$EF\left(t, g(t)\right) = g(t) = P\left(t, g(t)\right).$$

Since, $EF(t, X) \subset Q(t, X)$.
Therefore, there exist a measurable mapping $\xi': \Omega \to C$ such that

$$EF(t, g(t)) = Q(t, \xi'(t))$$

Therefore,

$$g(t) = EF\left(t, g(t)\right) = P(t, g(t)) = Q(t, \xi'(t))$$

Step III: For all $t \in \Omega$,

$$\|EF(t, g_{2n}(t)) - RS(t, \xi'(t))\|^2$$
$$\leq \alpha(t)\|P(t, g_{2n}(t)) - EF(t, g_{2n}(t))\|^2 + \beta(t)\left\|Q\left(t, \xi'(t)\right) - RS\left(t, \xi'(t)\right)\right\|^2$$
$$+ \gamma(t)\left\|P(t, g_{2n}(t)) - Q\left(t, \xi'(t)\right)\right\|^2$$

Taking $n \to \infty$, we have

$$\|g(t) - RS(t, \xi'(t))\|^2 \leq \alpha(t)\|g(t) - g(t)\|^2 + \beta(t)\|g(t) - RS(t, \xi'(t)\|^2 + \gamma(t)\|g(t) - g(t)\|^2$$
$$\Rightarrow \left(1 - \beta(t)\right)\|g(t) - RS(t, \xi'(t)\|^2 \leq 0$$
$$\Rightarrow RS\left(t, \xi'(t)\right) = g(t).$$

Hence

$$RS\left(t, \xi'(t)\right) = g(t) = Q\left(t, \xi'(t)\right).$$

Since, RS and Q are weakly compatible.
Therefore, for all $t \in \Omega$

$$RS\left(t, Q\left(t, \xi'(t)\right)\right) = Q\left(t, RS\left(t, \xi'(t)\right)\right);$$
$$RS\left(t, g(t)\right) = Q\left(t, g(t)\right).$$

ISSN: 2349- 5197
Impact Factor (PIF): 2.138

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Step IV: For all $t \in \Omega$,

$$\|EF(t,g(t)) - RS(t,g(t))\|^2 \leq \alpha(t)\|P(t,g(t)) - EF(t,g(t))\|^2 + \beta(t)\|Q(t,g(t)) - RS(t,g(t))\|^2 + \gamma(t)\|P(t,g(t)) - Q(t,g(t))\|^2$$

Taking $n \to \infty$, we have

$$\|g(t) - Q(t,g(t))\|^2 \leq \alpha(t)\|g(t) - g(t)\|^2 + \beta(t)\|Q(t,g(t)) - Q(t,g(t)\|^2 + \gamma(t)\|g(t) - Q(t,g(t))\|^2$$
$$\Rightarrow \left(1 - \gamma(t)\right)\|g(t) - Q(t,g(t)\|^2 \leq 0$$
$$\Rightarrow Q(t,g(t)) = g(t).$$

Thus for all $t \in \Omega$,

$$EF(t,g(t)) = RS(t,g(t)) = P(t,g(t)) = Q(t,g(t)) = g(t).$$

Hence $g(t)$ is a common random fixed point of EF, RS, P and Q.
Similarly, we can proof that for RS is continuous.
Uniqueness: Suppose that $h(t): \Omega \to C$ be the another common random fixed point of the random operators EF, RS, P and Q.
Therefore for all $t \in \Omega$,

$$EF(t,h(t)) = RS(t,h(t)) = P(t,h(t)) = Q(t,h(t)) = h(t).$$

$$\|g(t) - h(t)\|^2 = \|EF(t,g(t)) - RS(t,h(t))\|^2$$
$$\leq \alpha(t)\|P(t,g(t)) - EF(t,g(t))\|^2 + \beta(t)\|Q(t,h(t)) - RS(t,h(t))\|^2 + \gamma(t)\|P(t,g(t)) - Q(t,h(t))\|^2$$
$$\Rightarrow \|g(t) - h(t)\|^2 \leq \alpha(t)\|g(t) - g(t)\|^2 + \beta(t)\|h(t) - h(t)\|^2 + \gamma(t)\|g(t) - h(t)\|^2$$
$$\Rightarrow \|g(t) - h(t)\|^2 \leq \gamma(t)\|g(t) - h(t)\|^2$$
$$\Rightarrow g(t) = h(t).$$

Hence $g(t)$ is a unique common random fixed point of random operators EF, RS, P and Q.

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