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FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS

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Abstract
The main objective of the paper is to establish some common fixed point theorems for weakly reciprocally continuous in the realm of metric spaces .These results extended and improved several well known results, in particular, the result of pant et al,(2011), Giniswamy et al,(2012) and recent result of Giniswamy and Maheswari P.G.,(2014)..

Introduction
Jungck(1976,1986,1996) extended the concept of weakly commuting mappings (defined by Sessa,1982) to Compatible and then to weakly compatible mappings, which is widely used to prove common fixed point theorems. In 1998 Pant introduced the concept of reciprocal continuity of the mappings at the common fixed points .As a generalization of this, in 2011,Pant et al,(2011) defined the notion of weak reciprocal continuous mappings, which extended the scope of the study of common fixed point theorems from the class of compatible continuous mappings to a wider class of mappings that includes noncompatible and discontinuous mappings. And recent result of Giniswamy and Maheswari P.G.,(2014) defined the Fixed point theorems for reciprocally continuous mappings. In his paper He has proved some common fixed point theorems in metric space by using the concept of weakly reciprocally continuous self mappings of a metric space. Also ,we illustrate some results by using Compatible mappings. The following are the basic definitions needed in the main result.

Definition 1.1 Two self maps f and g of a metric space (X,d) are called compatible if $\lim_n d(fgx_n, gfx_n)$ =0, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n$ =t for some t in x. thus the mappings f and g will be noncompatible if there exits at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent.

Definition 1.2 Two self maps f and g of a metric space (X,d) are called R-weakly commuting on X. if there exists some positive real number R such that $d(fgx_n, gfx_n) \leq$ Rd(f_x, g_x) for all x in X.

Definition 1.3 Two self mappings f and g of a metric space (X,d) are called R-weakly commuting of type (A_g) if there exists some positive real number R such that $d(fgx_n, gfx_n) \leq$ Rd(f_x, g_x) for all x in X. Similarly two self mappings f and g of a metric space(X,d) are called R-weakly commuting of type (A_f) if there exists some positive real number R such that $d(fgx_n, gfx_n) \leq$ Rd(f_x, g_x) for all x in X.

Definition 1.4 Two self mappings f and g of a metric space (X,d) are called R-weakly commuting of type(P) if there exists some positive real number R such that $d(ffx, ggx) \leq$ Rd(f_x, g_x) for all x in X.

Definition 1.5 Two self mappings f and g of a metric space (X,d) are called Reciprocally continuous if $\lim_n fgx_n =$ ft and $\lim_n gfx_n$ =gt whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

Definition 1.6 Two self mappings f and g of a metric space (X,d) are called weakly Reciprocally continuous if $\lim_n fgx_n =$ ft or $\lim_n gfx_n$ =gt whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

Main Result
Theorem 2.1
Let f and g be two weakly reciprocally continuous self mappings of a metric space (X,d) such that
1. fX ⊆ gX and fX is complete.
2. d(fx,fy) ≤ ad(gx,gy) + bd(fx,fy) +c d(fy,gy) + e max{ d(gx,fx),d(fy,gy)} with 0 ≤ a,b,c,e < 1 and 0 ≤ a+b+c+2e < 1.

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If f and g are either compatible or R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f) or R-weakly commuting of type(P) then f and g have a unique common fixed point.

Proof: Let x_0 be any point in X. since fX ⊆ gX there exists a sequence of points x_0, x_1, x_2, ----- x_n such that x_{n+1} is in the preimage under g of fx_n

i.e. $fx_0 = gx_1$, . $fx_1 = gx_2$ --------. $fx_n = gx_{n+1}$

Also define a sequence $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for n= 0,1,2,3-----

Clearly $\{y_n\}$ is a sequence in fX . Now we claim that $\{y_n\}$ is a Cauchy sequence in fX.

Using (2) we get

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1})$$
$$\leq ad(gx_n, gx_{n+1}) + bd(fx_n, fx_{n+1}) + c\,d(fx_{n+1}, gx_{n+1}) + e\max\{d(gx_n, fx_n), d(fx_{n+1}, gx_{n+1})\}$$
$$= ad(y_{n-1}, y_n) + bd(y_n, y_{n+1}) + c\,d(y_{n+1}, y_n) + e\max\{d(y_{n-1}, y_n), d(y_{n+1}, y_n)\}$$

$$d(y_n, y_{n+1}) - bd(y_n, y_{n+1}) - c\,d(y_n, y_{n+1}) - e\,d(y_n, y_{n+1}) = ad(y_{n-1}, y_n) + e\,d(y_{n-1}, y_n)$$
$$d(y_n, y_{n+1})\,(1-b-c-e) = (a+e)\,d(y_{n-1}, y_n)$$
$$d(y_n, y_{n+1}) = \frac{(a+e)}{(1-b-c-e)}\,d(y_{n-1}, y_n)$$

i.e. $d(y_n, y_{n+1}) \leq K\,d(y_{n-1}, y_n) \leq K^n\,d(y_0, y_1)$

where $K = \frac{(a+e)}{(1-b-c-e)} < 1$

Also for every integer P>0 we have

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) \text{ ----------} + d(y_{n+p-1}, y_{n+p})$$
$$\leq K^n\,(1 + K + K^2 + \text{-------} + K^{p-1}\,d(y_0, y_1)$$
$$\leq \frac{1}{1-K}\,K^n\,d(y_0, y_1)$$

That is $d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. therefore $\{y_n\}$ is a Cauchy sequence in fX.

Since fX is complete ,there exists a point t∈ fX such that $y_n \to t$ as $n \to \infty$,where t = ft_1 for some t_1 in X . moreover $y_n = fx_n = gx_{n+1} \to t$.

Case(i) : Suppose that f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$.

Firstly , let $gfx_n \to gt$. Then compatibility of f and g gives $\lim_n (fx_n, gx_n) = 0$.

As $n \to \infty$ we get $fgx_n \to gt$. From (1) we obtain $fgx_{n+1} = ffx_n \to gt$. Using (2) we get

$$d(ft, ffx_n) \leq ad(gt, gfx_n) + bd(ft, ffx_n) + cd(ffx_n, gfx_n) + e\max\{d(gt, ft), d(ffx_n, gfx_n)\}$$

On letting $n \to \infty$ we get ft=gt , since b+e<1 . As compatibility implies commutativity at coincidence point, we obtain fft = fgt = gft = ggt. Using (2) we now get

$$d(ft, fft) \leq ad(gt, gft) + bd(ft, fft) + c\,d(fft, gft) + e\max\{d(gt, ft), d(fft, gft)\}$$

This implies ft= fft, since a+e<1. Hence ft = fft =gft and ft is a common fixed point of f and g.

Next, let $fgx_n \to ft.\, then$ fX ⊆ gX implies that ft=gu for some u ∈ X and hence $fgx_n \to gu$. Compatibility of f and g implies $gfx_n \to gu$. By using (1) we get $fgx_{n+1} = ffx_n \to gu$.

Using (2) we get

$$d(fu, ffx_n) \leq ad(gu, gfx_n) + bd(fu, ffx_n) + cd(ffx_n, gfx_n) + e\max\{d(gu, fu), d(ffx_n, gfx_n)\}$$

On letting $n \to \infty$ we get fu=gu , since b+e<1 . Again compatibility of f and g gives ffu = fgu = gfu Finally, Using (2)

$$d(fu, ffu) \leq ad(gu, gfu) + bd(fu, ffu) + c\,d(ffu, gfu) + e\max\{d(gu, fu), d(ffu, gfu)\}$$

which gives fu =ffu . Hence fu= ffu = gfu and fu is a common fixed point of f and g.

Case (ii): Now suppose that f and g are R-weakly commuting of type (A_g).

Weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$.

Firstly , let $gfx_n \to gt$. Then R-weak commutativity of type (A_g) of f and g gives

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d(ffx_n , gfx_n) ≤ R d(fx_n , gx_n).
As n→ ∞ we get $ffx_n \to gt$. Also using (2) we get
d(ft, ffx_n) ≤ ad(gt, gfx_n) +bd(ft, ffx_n)+c d(ffx_n, gfx_n)+e max{d(gt, ft), $d(ffx_n, gfx_n)$ }
On letting n→ ∞ we get ft=gt , since b+e<1 . R-weak Commutativity of type (A_g) implies d(fft, gft) ≤ R d(ft,gt) .This gives fft = gft and hence fft = fgt = gft = ggt using (2) we get
d(ft, fft) ≤ ad(gt, gft) +bd(ft, fft) + c d(fft, gft) + emax{d(gt, ft), $d(fft, gft)$ }
This implies ft= fft . Hence ft = fft =gft and ft is a common fixed point of f and g.
Next Suppose $fgx_n \to ft.then$ fX ⊆ gX implies that ft=gu for some u ∈ X and hence $fgx_n \to gu$.
By (1) this gives $ffx_n \to gu$. R- weak commutativity of type (A_g) implies d(ffx_n, gfx_n) ≤ $R\ d(fx_n, gx_n)$.
As n→ ∞ we get $gfx_n \to gu$. Now using (2) we have
d(fu, ffx_n) ≤ad(gu, gfx_n)+bd(fu, ffx_n)+cd(ffx_n, gfx_n)+ emax{d(gu, fu), $d(ffx_n, gfx_n)$}
letting n→ ∞ we get fu=gu, since b+e<1. Again R-weak commutativity of type (A_g) implies
d(ffu,gfu)≤ R d(fu,gu). This gives ffu= gfu and hence ffu = fgu = gfu = ggu.
Finally using (2) we get
d(fu, ffu) ≤ ad(gu, gfu) +bd(fu, ffu) + c d(ffu, gfu) + e max{d(gu, fu), $d(ffu, gfu)$}
thus fu= ffu since a+e<1. Hence fu=ffu= gfu and fu is a common fixed point of f and g.

Case (iii): Next Suppose that f and g are R-weakly Commuting of type (A_f).
Again, Weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$.
First suppose $gfx_n \to gt$ by virtue of (1) this gives $ggx_n \to gt$. Then R-weak commutativity of type (A_f) gives d(ffx_n, gfx_n) ≤ $R\ d(fx_n, gx_n)$.
As n→ ∞ we get $fgx_n \to gt$. Also using (2) we get
d (ft, fgx_n) ≤ ad(gt, ggx_n) +bd(ft, fgx_n) + c d(fgx_n, ggx_n) + e max{d(gt, ft), $d(fgx_n, ggx_n)$ }
On letting n→ ∞ we get ft=gt , since b+e<1 . R-weak Commutativity of type (A_f) implies d(fgt, ggt) ≤ R d(ft,gt) .This gives fgt = ggt and hence fft = fgt = gft = ggt using (2) we get
d(ft, fft) ≤ ad(gt, gft) +bd(ft, fft) + c d(fft, gft) + emax{d(gt, ft), $d(fft, gft)$ }
This implies ft= fft . Hence ft = fft =gft and ft is a common fixed point of f and g.
Next Suppose $fgx_n \to ft.then$ fX ⊆ gX implies that ft=gu for some u ∈ X and hence $fgx_n \to gu$.
R- weak commutativity of type (A_f) implies d(fgx_n, ggx_n) ≤ $R\ d(fx_n, gx_n)$.
As n→ ∞ we get $ggx_n \to gu$. Now using (2) we have
d(fu, fgx_n) ≤ ad(gu, ggx_n) +bd(fu, fgx_n) + cd(fgx_n, ggx_n) +emax{d(gu, fu), $d(fgx_n, ggx_n)$}
On letting n→ ∞ we get fu=gu, since b+e<1. Again R-weak commutativity of type (A_f) implies
d(fgu,ggu)≤ R d(fu,gu). This gives fgu= ggu and hence ffu = fgu = gfu = ggu.
Finally using (2) we get
d(fu, ffu) ≤ ad(gu, gfu) +bd(fu, ffu) + c d(ffu, gfu) + e max{d(gu, fu), $d(ffu, gfu)$}
Hence fu=ffu= gfu and fu is a common fixed point of f and g.

Case (iv): Finally, Suppose that f and g are R-weakly Commuting of type (P).
The Weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$.
First suppose $gfx_n \to gt$ by (1) we get $ggx_n \to gt$. Then R-weak commutativity of type (P) gives d(ffx_n, ggx_n) ≤ $R\ d(fx_n, gx_n)$.
As n→ ∞ we get $ffx_n \to gt$. Also using (2) we get
d(ft, ffx_n) ≤ ad(gt, gfx_n) +bd(ft, ffx_n) + c d(ffx_n, gfx_n) + e max{d(gt, ft), $d(ffx_n, gfx_n)$ }
On letting n→ ∞ we get ft=gt , since b+e<1 . R-weak Commutativity of type (P) implies d(fft, ggt) ≤ R d(ft,gt) .This gives fft = ggt and hence fft = fgt = gft = ggt using (2) we get
d(ft, fft) ≤ ad(gt, gft) +bd(ft, fft) + c d(fft, gft) + emax{d(gt, ft), $d(fft, gft)$ }

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This implies ft= fft . Hence ft = fft =gft and ft is a common fixed point of f and g.
Next Suppose $fgx_n \to ft$. This also implies $ffx_n \to ft$. Since fX ⊆ gX implies we have ft=gu for some u ∈ X and hence $ffx_n \to gu$. R- weak commutativity of type (P) implies d$(ffx_n, ggx_n) \leq R\, d(fx_n, gx_n)$.
Letting n→ ∞ we get $ggx_n \to gu$. Now using (2) we have
d(fu, fgx_n) ≤ ad(gu, ggx_n) +bd(fu, fgx_n) + cd(fgx_n, ggx_n) +emax{d$(gu, fu), d(fgx_n, ggx_n)$}
As n→ ∞ we get fu=gu, since b+e<1. Again R-weak commutativity of type (P) implies
d(ffu,ggu)≤ R d(fu,gu). This gives ffu= ggu and hence ffu = fgu = gfu = ggu.
Finally using (2) we get
d(fu, ffu) ≤ ad(gu, gfu) +bd(fu, ffu) + c d(ffu, gfu) + e max{d$(gu, fu), d(ffu, gfu)$}
Hence fu=ffu= gfu and fu is a common fixed point of f and g.
Uniqueness of the Common fixed point can be Proved easily in each of the four cases.

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References
1. Giniswamy and Maheswari P.G. (2014): Fixed point theorems for reciprocally continuous mappings, Bull. Cal. Math. Soc. , 106,(1) 19-30.
2. Giniswamy ,Jayanthi C. and Maheshwari,P.G. (2012): Fixed point theorems for compatible and noncompatible self maps,J. Int. Acad. Phys. Sci.,vol.16, no. 1,1.
3. Jungck, G.(1976): Commuting mappings and Fixed point,Amer.Math.Monthly,83,261.
4. Jungck,G.(1986): Compatible mappings and common fixed points,Int. J Math. Math.Sci.,9,771
5. Jungck ,G.(1996): Common fixed points for continuous nonself maps on non metric spaces, far East J. Math.sci.4,199.
6. Pant,R.P. (1998): Common fixed points for four mappings,Bull,Cal.Math.Soc.,90,281.
7. Pant , R.P.(1994): Common Fixed point of non-commuting mappings ,J. Math. Anal.Appl.188,436.
8. Pant, R.P; Bisht , R.K. and Arora, D.(2011): Weak reciprocal continuity and fixed point theorems, Ann. Univ. Ferrara ,57(1), 181.
9. Pathak, H.K. ; Cho, Y. J. and Kang, S.M. (1997): Remarks of R-weakly commuting mappings and common fixed points theorems, Bull. Korean Math. Soc. , 34,247.
10. Sessa, S. (1982): On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. , 32(46), 149.